

REDUCTION OF STRESS CONCENTRATION IN A PLATE OF
UNLIMITED DIMENSIONS WITH ANGULAR NOTCHES

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REDUCTION OF STRESS CONCENTRATION IN A PLATE OF
UNLIMITED DIMENSIONS WITH ANGULAR NOTCHES

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Reduction in stress concentration in plates with angular notches, by modifying the continuity of the notch without changing the radius of curvature, is attempted on the example of a plate of infinite dimensions with a quasi-square hole. For a plate extended along the Ox axis by uniformly distributed forces, the reduced contour conditions for the opening are mathematically derived, and the maximum stress along the contour is calculated. A reduction in maximum stress by 20% was obtained by optimizing the contour.

Author
In order to decrease the stress concentration in a plate with triangular and other angular notches, the designer usually rounds off all angles and attempts to increase the radius of curvature as much as possible. However, the maximum radius of curvature is dictated by design considerations and may be considerably restricted. It is also possible, at constant radius, to reduce the stress concentration by improving the continuation of the notch. The present work is devoted to this question.

Let us consider the problem as applied to extension along the axis Ox of an unbounded plate with an almost square opening. The problem of the distribution of stresses in a plate with a notch of various configurations has been solved by Academician N.I.Muskhelishvili (Bibl.1) and, for a number of important cases, by Academician G.N.Savin (Bibl.2).

Let us take as the representative function

$$z=\omega(\zeta)=B\left(\frac{1}{\zeta}-k\zeta^3+m\zeta^7-n\zeta^{11}\right). \quad (1)$$

The coefficients k, m, and n will be regarded as parameters. Let us find their values under the conditions that their minimum radius of curvature, corresponding to an angle of $\theta = 45^\circ$ ($\zeta = e^{i\theta}$), must be equal to a specified value ρ_{min} , that the maximum stress on the curve near the angle $\theta = 45^\circ$ is the stress at the point corresponding to $\theta = 90^\circ$, and that the maximum stress must be minimum.

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** Numbers in the margin indicate pagination in the original foreign text.

The radius of curvature $\rho_{s,1n}$ for the representative function (1) and the angle $\theta = 45^\circ$ will be

$$\varrho_{\min} = -B \times \frac{(1 - 3k - 7m - 11n)^3}{-27k^3 - 343m^2 - 1331n^2 + 6k - 210km - 1386mn + 42m - 462kn + 110n}. \quad (2)$$

The stress function $\varphi(\zeta)$ is represented in the form

$$\varphi(\zeta) = \varphi_1(\zeta) + \varphi_0(\zeta), \quad (3)$$

where $\varphi_1(\zeta)$ characterizes the stresses in a plate without an opening; $\varphi_0(\zeta)$ are the additional stresses connected with the presence of a notch. Here, $\varphi_0(\zeta)$ is determined from the functional equation of N.I. Muskhelishvili (Bibl.1)

$$\varphi_0(\zeta) + \frac{1}{2\pi i} \int_V \frac{\omega(\sigma)}{\bar{\omega}'(\sigma)} \overline{\varphi_0'(\sigma)} \frac{d\sigma}{\sigma - \zeta} + \bar{b}_0 = \frac{1}{2\pi i} \int_V \frac{f_1^0 + if_2^0}{\sigma - \zeta} d\sigma. \quad (4)$$

For a plate extended along the axis Ox by uniformly distributed forces p, the reduced contour conditions $f_1^0 + if_2^0$ and the function $\varphi_1(\zeta)$ are /135

$$f_1^0 + if_2^0 = -\frac{p}{2} [\omega(\sigma) - \overline{\omega(\sigma)}]; \quad (5)$$

$$\varphi_1(\zeta) = \frac{p}{4} \omega(\zeta). \quad (6)$$

Let us determine the integral on the left-hand side of eq.(4) for the representative function (1):

$$\frac{\omega(\zeta)}{\bar{\omega}'\left(\frac{1}{\bar{\zeta}}\right)} = c_9 \zeta^9 + c_8 \zeta^8 + c_1 \zeta + \sum_{k=1}^{\infty} c_{1-4k} \zeta^{1-4k}; \quad (7)$$

$$c_9 = n; \quad c_8 = -(m + 3kn); \quad c_1 = k + 7mn + 3k(m + 3kn);$$

$$\varphi_0(\zeta) = \sum_{n=1}^{\infty} a_n \zeta^n; \quad (8)$$

$$\begin{aligned} \frac{1}{2\pi i} \int_V \frac{\omega(\sigma)}{\bar{\omega}'(\sigma)} \overline{\varphi_0'(\sigma)} \frac{d\sigma}{\sigma - \zeta} = & \bar{a}_1 c_9 \zeta^9 + 2\bar{a}_2 c_9 \zeta^8 + 3\bar{a}_3 c_9 \zeta^7 + 4\bar{a}_4 c_9 \zeta^6 + \\ & + (5\bar{a}_5 c_9 + \bar{a}_1 c_8) \zeta^5 + (6\bar{a}_6 c_9 + 2\bar{a}_2 c_8) \zeta^4 + (7\bar{a}_7 c_9 + 3\bar{a}_3 c_8) \zeta^3 + \\ & + (8\bar{a}_8 c_9 + 4\bar{a}_4 c_8) \zeta^2 + (9\bar{a}_9 c_9 + 5\bar{a}_5 c_8 + \bar{a}_1 c_1) \zeta + 10\bar{a}_{10} c_9 + 5\bar{a}_6 c_8 + 2\bar{a}_2 c_1. \end{aligned} \quad (9)$$

The integral on the right-hand side of eq.(4) is equal to

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f_1^0 + if_2^0}{\sigma - \zeta} d\sigma = \frac{\rho B}{2} (\zeta + k\zeta^2 - m\zeta^7 + n\zeta^{11}). \quad (10)$$

After substitution of the resultant expressions into eq.(4) and after equating the coefficients of the same powers of ζ , we find that the even coefficients a_n are zero. Therefore,

$$\varphi_0(\zeta) = a_1\zeta + a_3\zeta^3 + a_5\zeta^5 + a_7\zeta^7 + a_9\zeta^9 + a_{11}\zeta^{11}, \quad (11)$$

where the coefficients sought are determined from the equations

$$\begin{aligned} \bar{a}_1 c_9 + a_9 &= 0; & 3\bar{a}_3 c_9 + a_7 &= -\frac{\rho B}{2} m; \\ 5\bar{a}_5 c_9 + \bar{a}_1 c_5 + a_5 &= 0; & 7\bar{a}_7 c_9 + 3\bar{a}_5 c_5 + a_3 &= \frac{\rho B}{2} k; \\ 9\bar{a}_9 c_9 + 5\bar{a}_7 c_5 + \bar{a}_1 c_1 + a_1 &= \frac{\rho B}{2}; & a_{11} &= \frac{\rho B}{2} n. \end{aligned} \quad (12)$$

Consequently,

$$\varphi(\zeta) = a_1\zeta + a_3\zeta^3 + a_5\zeta^5 + a_7\zeta^7 + a_9\zeta^9 + a_{11}\zeta^{11} + \frac{\rho}{4} \omega(\zeta). \quad (13)$$

On the contour of the opening, σ_0 equals

$$\sigma_0 = 4 \operatorname{Re} \Phi(\sigma), \quad (14)$$

where

$$\Phi(\sigma) = \frac{\varphi'(\sigma)}{\omega'(\sigma)} = \frac{\rho}{4} + \frac{\varphi_0'(\sigma)}{\omega'(\sigma)}. \quad (15)$$

This means that

$$\sigma_0 = \rho + 4 \operatorname{Re} \frac{\varphi_0'(\sigma)}{\omega'(\sigma)}. \quad (16)$$

For the representative function (1)*, we have

$$\begin{aligned} \sigma &= \rho + 4 \frac{A_0 + A_2 \cos 2\theta + A_4 \cos 4\theta + A_6 \cos 6\theta + A_8 \cos 8\theta + \\ &\quad + A_{10} \cos 10\theta + A_{12} \cos 12\theta}{B(D_0 + D_4 \cos 4\theta + D_8 \cos 8\theta + D_{12} \cos 12\theta)} = \\ &= \rho + 4 \frac{\alpha(k, m, n, \theta)}{B\beta(k, m, n, \theta)}, \end{aligned} \quad (17)$$

where $A_0, A_2, \dots, D_0, D_4, \dots$ are known functions with k, m , and n .

We will now attempt to satisfy the requirements posed. The angle θ corresponding to σ_{\max} , is determined from the condition $\frac{\partial \sigma}{\partial \theta} = 0$ which leads to the equality

$$\frac{\alpha'_\theta}{\beta'_\theta} = \frac{\alpha}{\beta}; \quad (18)$$

where α'_θ and β'_θ are derivatives of α and β with respect to θ .

The equality of the maximum stresses on the curvature and at the point marking $\theta = 90^\circ$ is written in the form

$$\sigma_{90} = \sigma_{\max} \quad \text{or} \quad \frac{-3a_3 + 7a_7 - 11a_{11}}{-1 + 3k + 7m + 11n} = \frac{\alpha(k, m, n, \theta_1)}{\beta(k, m, n, \theta_1)}. \quad (19)$$

Let us minimize the value of σ_{\max} *:

$$d\sigma = \frac{\partial \sigma}{\partial k} dk + \frac{\partial \sigma}{\partial m} dm + \frac{\partial \sigma}{\partial n} dn = 0. \quad (20)$$

If we bear in mind eq.(20) and the connectivity formula (2), we obtain

$$\frac{\partial Q}{\partial k} dk + \frac{\partial Q}{\partial m} dm + \frac{\partial Q}{\partial n} dn = 0; \quad (21)$$

$$\frac{\partial \sigma_{90}}{\partial k} dk + \frac{\partial \sigma_{90}}{\partial m} dm + \frac{\partial \sigma_{90}}{\partial n} dn = 0. \quad (22)$$

To find the minimum, in the presence of connectivity, we used the indeterminate Lagrange multipliers. In the special case under consideration, however, when the number of connectivity conditions is one less than the number of parameters, we can determine the minimum without making use of Lagrange multipliers. From eqs.(21) and (22) we find the values of $\frac{dm}{dk}$ and $\frac{dn}{dk}$ and substitute them into eq.(20). The transformations reduce the determinant to zero

$$\begin{vmatrix} \frac{\partial \sigma}{\partial k} & \frac{\partial Q}{\partial k} & \frac{\partial \sigma_{90}}{\partial k} \\ \frac{\partial \sigma}{\partial m} & \frac{\partial Q}{\partial m} & \frac{\partial \sigma_{90}}{\partial m} \\ \frac{\partial \sigma}{\partial n} & \frac{\partial Q}{\partial n} & \frac{\partial \sigma_{90}}{\partial n} \end{vmatrix} = 0. \quad (23) \quad \text{137}$$

* Hereafter, the subscripts θ and \max after σ will be omitted.

Equations (2), (18), (19), and (23) will yield the required values of k , m , n , and θ .

This problem involves a large computational effort, so that we solved a simpler problem as a typical example. The representative function was taken in the form

$$z = \omega(\zeta) = B \left(\frac{1}{\zeta} - k\zeta^3 + m\zeta^7 \right). \quad (24)$$

In determining the parameters k and m , we stipulated that the first two conditions be satisfied, namely, a prescribed ρ_{min} and equality of the greatest stresses near $\theta = 45^\circ$ and the stress at $\theta = 90^\circ$. Thus, the parameters k and m and the angle θ , corresponding to the maximum stress, can be determined from eqs.(2), (18), and (19). For a prescribed ρ_{min} , the following was taken into account: The representative function for an infinite plate with the square opening has the form (Bibl.2)

$$z = \omega(\zeta) = B \left(\frac{1}{\zeta} - \frac{1}{6}\zeta^3 + \frac{1}{56}\zeta^7 - \frac{1}{176}\zeta^{11} + \dots \right). \quad (25)$$

Confining the calculation to the first two terms of the series and assuming that

$$z = \omega(\zeta) = B \left(\frac{1}{\zeta} - \frac{1}{6}\zeta^3 \right), \quad (26)$$

we obtain $\rho_{min} = 0.1B = 0.06a$, where a is the length of a side of the square. The same value $\rho_{min} = 0.1B$ was also adopted for our example.

The required quantities were found to be: $k = 0.074$, $m = 0.0307$, $\theta = 48^\circ$.

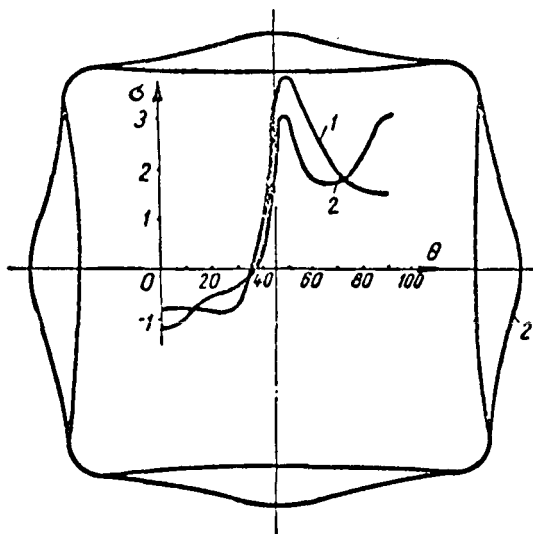
The sketch shows the configuration of the notches and the stress distribution over the contour, plotted against the angle θ for the interval $0 + 90^\circ$ with reference to the representative functions (26) and (24) and denoted, respectively, by 1 and 2. The maximum stresses were decreased from 3.86p on the contour 1 to 3.07p on the contour 2, or by 20%.

Similar auxiliary calculations show that, for $\rho_{min} = 0.2B$ ($k = 0.074$, $m = 0.0136$) the maximum stress is decreased to 2.52p. Consequently, σ_{max} is found to be even smaller (by 16%) than for the circle into which the contour of a square is transformed with increasing radius and for which $\sigma_{max} = 3p$. Consequently, this result is of particular interest. We note that $\rho_{min} = 0.2B$ is a frequently encountered radius.

In this connection we note the following two facts:

A. When a unit circle on an infinite plane with a notch of function (25) is used for the representation, the maximum radius of curvature at the point

$\theta = 45^\circ$ will be $0.1B$, if the calculation, as already mentioned, is confined to the first two terms of eq.(26), while with increasing number of terms, the value of the radius varies discontinuously ($0.1B$; $0.0417B$; $0.0242B$; ...). In



practical application, we meet notches for which ρ_{min} is considerably greater than $0.1B$ and notches for which ρ_{min} does not fit into the quantities that can be obtained by replacing the series (25) by a polynomial. Therefore, from /138 this point of view, it is desirable to introduce parameters permitting representations with a prescribed radius of curvature.

B. In the example considered here, we used only a single condition with respect to the stresses, namely, that the stress near an angle of 45° be reduced relative to the stress at an angle of $\theta = 90^\circ$. If we satisfy the second requirement, i.e., the minimization of σ_{max} , and if we introduce a larger number of parameters, then it can be assumed that the stresses will be still further reduced.

BIBLIOGRAPHY

1. Muskhelishvili, N.I.: Some Fundamental Problems of the Mathematical Theory of Elasticity (Nekotoryye osnovnyye zadachi matematicheskoy teorii uprugosti). Izd. Akad. Nauk SSSR, 1949.
2. Savin, G.N.: Stress Concentration around Openings (Kontsentratsiya napryazheniy okolo otverstiy). Gostekhizdat, 1951.